McGill University

Final Examination

Math 263

Differential Equations and Linear Algebra

21 April, 2010 2 pm - 5 pm

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Instructions

This exam consists of 8 questions.

Answer all of them.

Calculators are not permitted.

Text and formula sheets are not allowed.

Please answer in exam booklets.

This exam comprises the cover, 2 pages of questions and one table of Laplace transforms.

Differential Equations & Linear Algebra MATH 263 Final Exam

1. (10 marks) Find all solutions of the ordinary differential equation $(y')^4 = (y^4)'$

2. (10 marks) Solve the following ODE over the range $0 < t < \pi/2$

$$y' + \frac{1}{2}\frac{\cos t}{\sin t}\, y = \frac{1}{2\cos t}\, y^{-1}.$$

3. (10 marks) Find the solution of the equation

$$y'' - 6y' + 9y = x(1 + 2e^{3x})$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$.

4. (10 marks) Solve the ODE

$$\frac{dy}{dx} = \frac{y^3 + 3x^2 y}{3x^3}.$$

5. (10 marks) Use Laplace transforms (see attached table) to solve the equation

$$y'' + 4y = g(t),$$

where $y(0) = y'(0) = 0$ and

$$g(t) = \begin{cases} 0 & 0 \le t < 1, \\ 3(t-1) & 1 \le t < 2, \\ 3 & 2 \le t. \end{cases}$$

6. (10 marks) Use Laplace transforms (see attached table) to solve the equation $y'' + 2y' + y = f(t)$, where $y(0) = -3$, $y'(0) = 3$ and $f(t)$ has a well-behaved Laplace transform.

7. (10 marks) Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 3 & 0 \\ 5 & 3 \end{pmatrix}$$

(a) Do they have inverses?

(b) Calculate their eigenvalues.

(c) Calculate their eigenvectors

(d) What is their rank? Are they defective or not? What is the dimension of each eigenspace?

8. (10 marks)

(a) Refer to the results in question 7 to solve the following set of differential equations

$$y_1' = y_1 + 2y_2 - y_3$$

$$y_2' = y_1 + y_3$$

$$y_3' = 4y_1 - 4y_2 + 5y_3.$$

Which of the following describes the behaviour of the solutions as $x \to \infty$: i) all are bounded; ii) some are bounded; iii) all are unbounded?

(b) Now solve the nonhomogeneous set of equations

$$y_1' = y_1 + 2y_2 - y_3$$

$$y_2' = y_1 + y_3$$

$$y_3' = 4y_1 - 4y_2 + 5y_3 + 8e^{5t},$$

given that $y_1(0) = -1$ and $y_2(0) = y_3(0) = 0$.

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\dfrac{1}{s}, \quad s > 0$
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3. t^n; n = positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4. t^p, $p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}$, n = positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$